

Connections between the Dirichlet values and the Dutta-Ray solution for convex games

Alejandro Bernárdez Ferradás¹, Estela Sánchez Rodríguez¹ and Miguel Ángel Mirás Calvo²

¹SiDOR. Universidade de Vigo. Departamento de Estatística e Investigación Operativa. CITMAga

²RGEAF. Universidade de Vigo. Departamento de Matemáticas

ABSTRACT

We focus on the class of convex (or concave) games whose core is full-dimensional and is contained in the unit simplex. We show that, for games in this class, the core-center solution selects the expected value of the conditional flat Dirichlet distribution on the core, and that the Dutta-Ray solution recommends the mode of a conditional Dirichlet distribution, with equal parameters, given that it belongs to the core. Furthermore, to any game without cloned players (a special type of symmetric players), we can associate a new game that has k -clones of each player. Our main result establishes that the core-center solution converges under replication to the Dutta-Ray solution, that is, the aggregate value that the core-center assigns to each group of clones in the k -replica of a game converges to the value assigned by the Dutta-Ray solution to the group representative in the initial game.

Keywords: Convex games, Dutta-Ray solution, core-center solution, Dirichlet values, cloned players, convergence under replication.

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EXTRACT

The $(n - 1)$ -dimensional unit simplex is the set of all possible divisions of a unit among n agents. The Dirichlet distribution is a family of multiparametric probability distributions over the unit simplex. Then, a Dirichlet distribution, with n positive parameters, is a probability distribution over the set of divisions of one unit among n agents. In particular, the flat Dirichlet distribution, the member of the family with all parameters equal to 1, corresponds to the uniform distribution over the unit simplex.

Bernárdez et al. (2025) define the Dirichlet value, with n positive parameters, for a positive balanced game as the expected value of the conditional distribution of a Dirichlet distribution, with the given parameters, over the (unit-scaled) core of the game. A notable member of this family is the core-center solution (González and Sánchez (2007)), which corresponds to the flat Dirichlet distribution.

Here, we focus on the class of convex (or concave) games whose (unit-scaled) core is full-dimensional and is contained in the unit simplex. We show that, the mode of the conditional distribution of a Dirichlet distribution, with all parameters equal to $k > 1$, over the (unit-scaled) core of the game coincides with the allocation selected for that game by the Dutta-Ray solution (Dutta and Ray (1989)).

Next, we consider two special types of symmetric players. Two players are clones of each other in a game if, given any coalition to which one of them belongs, exchanging this player for his clone does not change the worth of the coalition. This concept was introduced in the context of cost games, with the name of marionettes, by Zumsteg (1995). On the other hand, two players are dual clones in a game if given any coalition that does not contain one of them, adding this player to the coalition and removing the other, if in the group, does not change the value of the coalition. Given a natural number k , we can associate to each (cost) game without cloned players a new game, its k -replica, that has exactly k -clones of each of its players. We know that the aggregate sum of the allocations assigned by the core-center to the cloned players of each group in the k -replica coincides with the allocation selected by the Dirichlet value, with all parameters equal to k , to the player representing that group in the original game. Our main result establishes that, as k increases to infinity, the amount that the core-center solution assigns to each group

of clones converges to the allocation recommended by the Dutta-Ray solution to the representative in the game without clones. In that sense we say that the core-center solution converges under replication to the Dutta-Ray solution. A similar result holds, with the appropriate modifications, for dual clones and games of gains or benefits.

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